# Mathematical Techniques

1. (*a*) Obtain partial fractions for

$$\frac{x}{x^2-1}, \quad x > 1.$$ **2**

(*b*) Use the result of (*a*) to find

$$\int \frac{x^3}{x^2-1}\,dx, \quad x > 1.$$ **4**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC2 |
|---|---|---|---|---|---|---|
| (*a*) | 2 | C | CN | A6 | $\frac{1}{2(x+1)} + \frac{1}{2(x-1)}$ | 2001 A5 |
| (*b*) | 4 | A/B | CN | I11 | $\frac{1}{2}x^2 + \frac{1}{2}\ln\lvert x^2-1\rvert + c$ | |

*[No marking instructions available]*

2. Express $\dfrac{1}{x^2-x-6}$ in partial fractions. **2**

Evaluate $\displaystyle\int_0^1 \frac{1}{x^2-x-6}\,dx$. **4**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC2 |
|---|---|---|---|---|---|---|
| (1) | 2 | C | CN | A6 | $\frac{1}{5}\frac{1}{x-3} - \frac{1}{5}\frac{1}{x+2}$ | 2004 A5 |
| (2) | 4 | C | CN | I11 | $\frac{1}{5}\ln\frac{4}{9}$ | |

[SQA] 3. Express in partial fractions

$$\frac{11-2x}{x^2+x-2}.$$ **3**

Hence obtain

$$\int_3^5 \frac{11-2x}{x^2+x-2}\,dx.$$ **3**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC2 |
|---|---|---|---|---|---|---|
| (1) | 3 | C | CN | A6 | | 2000 SY1 Q10 |
| (2) | 3 | A/B | CN | I11 | | |

$$\frac{11-2x}{x^2+x-2} = \frac{11-2x}{(x-1)(x+2)}$$

$$= \frac{A}{x-1} + \frac{B}{x+2}$$

$$11-2x = A(x+2) + B(x-1)$$ **1**

$$x = 1 \Rightarrow 3A = 9;\ A = 3$$ **1**

$$x = -2 \Rightarrow -3B = 15;\ B = -5$$ **1**

$$\int_3^5 \frac{11-2x}{x^2+x-2}dx = \int_3^5 \frac{3}{x-1} - \frac{5}{x+2}dx$$ **1**

$$= [3\ln(x-1) - 5\ln(x+2)]_3^5$$ **1**

$$= 3\ln 4 - 3\ln 2 - 5\ln 7 + 5\ln 5$$ **1**

$$= \left(\ln\frac{2^3 5^5}{7^5}\right) \approx 0.397.$$

4. (*a*) Given $f(x) = x^3 \tan 2x$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. **3**

(*b*) For $y = \dfrac{1+x^2}{1+x}$, where $x \neq -1$, determine $\dfrac{dy}{dx}$ in its simplest form. **3**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC2 |
|---|---|---|---|---|---|---|
| (*a*) | 3 | C | CN | D4, D2 | | 2005 Q1 |
| (*b*) | 3 | C | CN | D4 | | |

[SQA] 5. Differentiate the following functions with respect to $x$, simplifying your answers where possible.

(*a*) $h(x) = \sin(x^2)\cos(3x)$. **3**

(*b*) $y = \dfrac{\ln(x+3)}{x+3}$, $x > -3$. **3**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC2 |
|---|---|---|---|---|---|---|
| (*a*) | 3 | C | CN | D4, D3, D6, D2 | | 1999 SY1 Q3 |
| (*b*) | 3 | C | CN | D5, D8 | | |

**(a)**

$$h(x) = \sin(x^2)\cos(3x)$$ **1 method mark (product rule)**

$$h'(x) = 2x\cos(x^2)\cos(3x) - 3\sin(x^2)\sin(3x)$$ **1 for first term**
**1 for second term**

**(b)**

$$y = \frac{\ln(x+3)}{(x+3)}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x+3}(x+3) - \ln(x+3).1}{(x+3)^2}$$ **1 method (quotient)**
**1 for accuracy**

$$= \frac{1 - \ln(x+3)}{(x+3)^2}$$ **1 for simplifying**

[SQA] 6. Differentiate

$$g(x) = \frac{\sin x}{1+\cos x}, \quad -\pi < x < \pi,$$

and simplify your answer. **3**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC2 |
|---|---|---|---|---|---|---|
| | 3 | C | CN | D5, D2 | | 1998 SY1 Q1 |

$$g(x) = \frac{\sin x}{1+\cos x}$$

$$g'(x) = \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$ **1 method (quotient)**

**1 for accuracy**

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$ **1 for simplifying**

$$= \frac{1}{1+\cos x}$$

[SQA] 7. Use calculus to find all the values of $x$ for which the function

$$f(x) = (1+x)^2 e^{-x}, \quad x \in \mathbb{R},$$

is increasing. **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC2 |
|---|---|---|---|---|---|---|
| | 5 | A/B | CN | D4, D8, D3, | | 1998 SY1 Q6 |

$$f(x) = (1+x)^2 e^{-x}$$

$$f'(x) = 2(1+x)e^{-x} - (1+x)^2 e^{-x}$$ **1 method (product)**

**1 for accuracy**

$> 0$ for $f(x)$ to be increasing. **1**

$$(1+x)\left[2-(1+x)\right]e^{-x} > 0$$

$$(1+x)(1-x)e^{-x} > 0$$ **1 for factorising**

$x^2 < 1 \quad \Rightarrow -1 < x < 1$ **1**

**($-1 \leqslant x \leqslant 1$ is acceptable)**

[SQA] 8.

(*a*) Differentiate $f(x) = e^{x^2+3}$ with respect to $x$. **2**

(*b*) Differentiate $g(x) = \ln\sqrt{x^2+3}$ with respect to $x$. **3**

Hence find $\displaystyle\int \frac{5x}{x^2+3}\,dx$. **2**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC2 |
|---|---|---|---|---|---|---|
| (*a*) | 2 | C | CN | D8, D3, D2 | | 2000 SY1 Q1 |
| (*b1*) | 3 | A/B | CN | D8, D3, D6 | | |
| (*b2*) | 2 | A/B | CN | Higher | | |

**(a)**

$f(x) = e^{x^2+3} \Rightarrow f'(x) = 2x\,e^{x^2+3}$ **1 for $2x$**

**1 for $e^{x^2+3}$**

**(b) Method 1**

$g(x) = \ln\sqrt{x^2+3} = \frac{1}{2}\ln(x^2+3)$ **1 for simplifying**

$g'(x) = \dfrac{1}{2}\,\dfrac{2x}{(x^2+3)} = \dfrac{x}{(x^2+3)}.$ **1 for $2x$**

**1 for $(x^2+3)$**

**Method 2**

$g(x) = \ln\sqrt{x^2+3} = \ln(x^2+3)^{1/2}$

$g'(x) = \dfrac{\frac{1}{2}(x^2+3)^{-\frac{1}{2}}\,2x}{(x^2+3)^{\frac{1}{2}}} = \dfrac{x}{(x^2+3)}.$ $\left\{\begin{array}{r}\textbf{1 for } \frac{1}{2}(x^2+3)^{-\frac{1}{2}}\\ \textbf{1 for } 2x\\ \textbf{1 for } (x^2+3)^{\frac{1}{2}}\end{array}\right.$

**(c)**

$\displaystyle\int \frac{5x}{(x^2+3)}\,dx = 5\int \frac{x}{(x^2+3)}\,dx$ **1 for handling the '5'**

$= 5\ln\sqrt{x^2+3} + c$ **1 for result**

9. A curve has equation $xy + y^2 = 2$.

(*a*) Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of $x$ and $y$. **3**

(*b*) Hence find an equation of the tangent to the curve at the point $(1, 1)$. **2**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC1 |
|---|---|---|---|---|---|---|
| (*a*) | 3 | C | CN | D15 | $-\frac{y}{x+2y}$ | 2001 A7 |
| (*b*) | 2 | C | CN | Higher | $x + 3y - 4 = 0$ | |

*[No marking instructions available]*

10. A curve is defined by the parametric equations

$$x = t^2 + t - 1,\ y = 2t^2 - t + 2$$

for all $t$. Show that the point A$(-1, -5)$ lies on the curve and obtain an equation of the tangent to the curve at the point A. **6**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC1 |
|---|---|---|---|---|---|---|
| | 6 | C | CN | D17, Higher | $y = 5x + 10$ | 2002 A3 |

11. Use the substitution $x = 1 + \sin\theta$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)^3}\, d\theta$. **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC3 |
|---|---|---|---|---|---|---|
| | 5 | C | CN | I5 | $\frac{3}{8}$ | 2003 A5 |

| | |
|---|---|
| $\bullet^1$ differentiate substitution | $\bullet^1$ $dx = \cos\theta\, d\theta$ |
| $\bullet^2$ convert limits | $\bullet^2$ $x = 1, 2$ |
| $\bullet^3$ process substitution | $\bullet^3$ $\int_1^2 \frac{1}{x^3}\, dx$ |
| $\bullet^4$ integrate | $\bullet^4$ $\frac{x^{-2}}{-2}$ |
| $\bullet^5$ complete | $\bullet^5$ $\frac{-1}{8} - \frac{-1}{2}$ |

12. Use the substitution $u = 1 + x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$. **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC3 |
|---|---|---|---|---|---|---|
| | 5 | C | CN | I5 | | 2005 Q5 |

[SQA] 13. By means of the substitution $u = x^2 - 8$, find

$$\int_3^4 x^3\left(x^2-8\right)^{\frac{1}{3}}\,dx.$$ **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC3 |
|---|---|---|---|---|---|---|
| | 5 | C | CN | I5 | | 1997 SY1 Q5 |

$u = x^2 - 8 \Rightarrow du = 2x\,dx \Rightarrow dx = ½du$ **1**

$x = 3 \Rightarrow u = 1$ and $x = 4 \Rightarrow u = 8$ **1**

$\int_3^4 x^3(x^2-8)^{1/3}dx = \int_1^8 (u+8)(u)^{1/3}½du$ **1**

$= ½\int_1^8 \left(u^{4/3} + 8u^{1/3}\right)du$

$= ½\left[\frac{3}{7}u^{7/3} + 6u^{4/3}\right]_1^8$ **1**

$= ½\left[\frac{384}{7} + 96\right] - ½\left[\frac{3}{7} + 6\right]$ **1**

$= ½\left[54\frac{3}{7} + 90\right] = 72\frac{3}{14}$

14. Find the value of $\int_0^{\frac{\pi}{4}} 2x \sin 4x \, dx$. **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC3 |
|---|---|---|---|---|---|---|
| | 5 | C | CN | I12 | $\frac{\pi}{8}$ | 2001 A3 |

*[No marking instructions available]*

[SQA] 15.

(*a*) By using the substitution $u = 2\sin x$, or otherwise, evaluate the definite integral

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + 4\sin^2 x} \, dx.$$ **4**

(*b*) Use integration by parts to find

$$\int x^2 \ln x \, dx.$$ **3**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC2 |
|---|---|---|---|---|---|---|
| (*a*) | 4 | C | CN | I5, I10 | | 1996 SY1 Q8 |
| (*b*) | 3 | C | CN | I12 | | |

**(a)** $u = 2\sin x;\ du = 2\cos x \, dx$ **1**

$x = 0 \Rightarrow u = 0;\ x = \frac{\pi}{6} \Rightarrow u = 1$ **1**

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + 4\sin^2 x} \, dx &= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{1 + u^2} \, du \\ &= \tfrac{1}{2}\left[\tan^{-1} u\right]_0^1 \\ &= \frac{\pi}{8}.\end{aligned}$$

**1**

**1**

**(b)** $\int x^2 \ln x \, dx = \ln x \int x^2 \, dx - \int \frac{1}{x}\left(\frac{1}{3}x^3\right) dx$ **1**

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c.$ **1 for first term**

**1 for second term**

[SQA] 16. Use integration by parts to obtain

$$\int_0^3 x\sqrt{x+1}\,dx.$$ **6**

| Part | Marks | Level | Calc. | Content | Answer | U2 OC2 |
|---|---|---|---|---|---|---|
| | 6 | C | CN | I12, Higher | | 1999 SY1 Q9 |

$$\int_0^3 x(x+1)^{\frac{1}{2}}dx = \left[x\int(x+1)^{\frac{1}{2}}dx\right]_0^3 - \int_0^3 1.\tfrac{2}{3}(x+1)^{\frac{3}{2}}dx$$ **1 method (parts)**

**1 for attempting $\int(x+1)^{\frac{1}{2}}dx$**

$$= \left[x.\tfrac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^3 - \tfrac{2}{3}.\tfrac{2}{5}\left[(x+1)^{\frac{5}{2}}\right]_0^3$$ **1 for $\int(x+1)^{\frac{1}{2}}dx$**

**1 for $\int(x+1)^{\frac{3}{2}}dx$**

$$= \left(2\times 4^{\frac{3}{2}} - 0\right) - \frac{4}{15}\left(4^{\frac{5}{2}} - 1\right)$$ **1**

$$= 16 - 8\,\frac{4}{15}$$ **1**

$$= 7\,\frac{11}{15}$$

17. A solid is formed by rotating the curve $y = e^{-2x}$ between $x = 0$ and $x = 1$ through $360°$ about the $x$-axis. Calculate the volume of the solid that is formed. **5**

| Part | Marks | Level | Calc. | Content | Answer | U1 OC3 |
|---|---|---|---|---|---|---|
| | 5 | B | CN | I8 | $\frac{\pi}{4}(1-\frac{1}{e^4})$ | 2004 A11 |

18. Find the general solution of the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = x, \quad x > 0.$$ **4**

| Part | Marks | Level | Calc. | Content | Answer | U3 OC4 |
|---|---|---|---|---|---|---|
| | 4 | C | CN | DE5 | $y = \frac{1}{3}x^2 + \frac{c}{x}$ | 2001 B2 |

*[No marking instructions available]*

*[END OF QUESTIONS]*